Collective Phenomena Emerging from the Interactions Between Dynamical Processes in Multiplex Networks

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Networks are a powerful way to model and study a wide variety of complex phenomena [1]. In the recent years, the study of collective dynamical processes on complex networks has improved our understanding of many complex systems and shed light on a wide range of physical, biological and social phenomena including synchronization, disease spreading, transport and cascades. Of particular interest in these works is the interplay between the structure of the network and its dynamics [2]. In fact, the topology of a network has an effect on the dynamical processes that take place over the network, while some properties of the dynamics can reveal important information on the interaction network [3, 4]. Understanding the relations between structure and dynamics can provide a solid foundation for modeling, predicting, and controlling dynamical processes in the real world. However, save for a few notable exceptions, the majority of the studies so far have considered a single process on a single network, ignoring a very important ingredient: often the components of a complex system interact through two or more dynamics at the same time, and these dynamics usually depend on each other in highly non-trivial ways.

In this work we propose a general framework for modelling, through a multiplex network, the *coupling of dynamical processes* of the same type (e.g. the spreading of two coupled diseases) or of different types (for instance a synchronization dynamics and a diffusion process). Moreover, we demonstrate with a specific example that this coupling mechanism can give rise to the emergence of complex phenomena generated by the interactions between the different dynamical processes.

The natural way to consider M interacting dynamical processes taking place over a complex system is to use a multiplex network with M layers. Each layer contains the same number of nodes, N, and there exists a one-to-one correspondence between nodes in different layers, but the topology and the very same nature of the connections at each layer may be different. We then assign a different dynamical process to each layer. Considering for simplicity the case $M = 2$, we assume that the dynamics of the entire system is governed by the following equations:

$$\begin{cases} \dot{x}_i = F_{\omega_i}(\mathbf{x}, A^{[1]}) \\ \dot{y}_i = G_{\chi_i}(\mathbf{y}, A^{[2]}) \end{cases} \qquad i = 1, 2, \ldots N \qquad (1)$$

where $\mathbf{x} = \{x_1, x_2, \ldots, x_N\} \in \mathcal{R}^{\mathcal{N}}$ and $\mathbf{y} = \{y_1, y_2, \ldots, y_N\} \in \mathcal{R}^{\mathcal{N}}$ denote the states of the two dynamical processes, while the topologies of the two layers are encoded in the adjacency matrices $A^{[1]} = \{a_{ij}^{[1]}\}$ and $A^{[2]} = \{a_{ij}^{[2]}\}$ respectively, such that $a_{ij}^{[1]} = 1$ ($a_{ij}^{[2]} = 1$) if a link exists between nodes i and j in the first (second) layer, and $a_{ij}^{[1]} = 0$ ($a_{ij}^{[2]} = 0$) otherwise. The dynamical evolution of the two network processes is ruled respectively by the functions F_ω and G_χ, which depend on the sets of parameters ω and χ, so that the state x_i (y_i) of node i at the first (second) layer is a function of the state $\mathbf{x}$ ($\mathbf{y}$) and of the topology $A^{[1]}$ ($A^{[2]}$) of the first (second) layer. The key ingredient that connects the two dynamical processes is provided by the nature of the correspondence *between layers*. In fact, the parameter ω_i in function F_{ω_i} at layer 1 is itself a function of time which depends on the dynamical state y_i of node i at layer 2, while the parameter χ_i at layer 2 depends on the state x_i of node i at layer 1. Namely, we have:

$$\begin{cases} \dot{\omega}_i = f(\omega_i, y_i) \\ \dot{\chi}_i = g(\chi_i, x_i) \end{cases} \qquad i = 1, 2, \ldots N \qquad (2)$$

where f and g are two assigned functions.

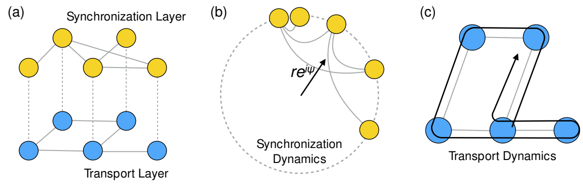

Figure 1: **Intertwined dynamical processes.** (a) An example of a two-layer multiplex of $N = 5$ nodes with neural synchronization dynamics at layer 1 (top), and transport dynamics at layer 2 (bottom). (b) The neural activity is described by the Kuramoto model, and the degree of synchronization is measured by the order parameter r. (c) The transport dynamics is modelled by biased random walkers. The two dynamical processes are bidirectionally coupled, as the natural frequencies of the oscillators at layer 1 depend on the distribution of random walkers at layer 2 and, at the same time, the random walkers are biased on the degree of synchronization of the nodes at layer 1.

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